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ON THE MOTION OF A SOLID BODY WITH ELASTIC AND DISSIPATIVE ELEMENTS

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F. L. CHERNOUS'KO
(Moscow)

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The general problem of dynamics of a solid body with internal degrees of freedom —linear elastic and dissipative elements —is analyzed. It is assumed that the periods of natural elastic oscillations and the time of their damping are small in comparison with the characteristic time of motion of the body relative to its center of mass. Approximate solutions are derived for internal degrees of freedom. General equations of motion of the system are obtained in the form of equations of solid body dynamics which contain additional terms that are due to inner elasticity and dissipation. The structure of these terms is determined. It is shown that in the case of a free system they consist of homogeneous polynomials of the fourth and fifth powers of components of the body angular velocity vector.

Problems of dynamics of a solid body containing elastic and dissipative elements were considered in a number of publications, for example [1-6].

1. Let us consider the motion of system S consisting of a solid body G of mass m and of N particles P_i of mass m_i each, $i = 1, \ldots, N$. Particles (further referred to as points) P_i are connected to the body and to each other by perfect elastic links with linear damping. The equilibrium position of points P_i relative to body G in the complete system at rest is denoted by O_i.

We introduce three Cartesian systems of coordinates: the stationary system $O'X'_1X_2'X_3'$, system $Ox_1x_2x_3$ rigidly attached to the solid body, and system $OX_1X_2X_3$ whose origin O is attached to the solid body and whose axes are parallel to the axes of system $O'X_1'X_2'X_3'$. We use the following notation:

$$\mathbf{R}_0 = O'O, \quad \rho_i = OO_i, \quad \mathbf{r}_i = O_iP_i \tag{1.1}$$

$$\mathbf{R}_i = O'P_i = \mathbf{R}_0 + \rho_i + \mathbf{r}_i, \quad i = 1, \ldots, N$$

The derivatives of scalar quantities with respect to time t are denoted by dots, and the derivatives of the three-dimensional vector $\mathbf{p}$ with respect to time in the coordinate systems $Ox_1x_2x_3$ and $O'X_1'X_2'X_3'$ are denoted, respectively, by $\mathbf{p}'$ and $\mathbf{p}^{\cdot}$. We have

$$\mathbf{p}^{\cdot} = \mathbf{p}' + \omega \times \mathbf{p} \tag{1.2}$$

where ω is the vector of absolute angular velocity of body G, i.e. of the coordinate system $Ox_1x_2x_3$. Note that, since points O and O_i are rigidly attached to body G, $\rho_i' = 0$, $i = 1, \ldots, N$, and evidently $\omega^{\cdot} = \omega'$.

Let us formulate the equations of motion of system S. We begin by considering the motion of points P_i relative to body G. We assume that the totality of points

P_i has n degrees of freedom relative to that body, and that the position of the totality in the coordinate system $Ox_1x_2x_3$ can be defined by the n-dimensional vector q of the generalized coordinates $q_1, \ldots, q_n$. We further assume that in the case of small oscillations the displacement vectors r_i linearly depend on the generalized coordinates

$$\mathbf{r}_i = \sum_{i=1}^{n} \mathbf{H}_{ij} q_i, \quad i = 1, \ldots, N \tag{1.3}$$

where $\mathbf{H}_{ij}$ are constant vectors in the coordinate system $Ox_1x_2x_3$. The kinetic energy of the motion of points P_i relative to that coordinate system with allowance for equality (1.3) is of the form

$$\frac{1}{2} \sum_{i=1}^{N} m_i (\mathbf{r}_i')^2 = \frac{1}{2} \sum_{j,\,k=1}^{n} a_{jk} q_j^{\cdot} q_k^{\cdot} \tag{1.4}$$

$$a_{jk} = \sum_{i=1}^{N} m_i \mathbf{H}_{ij} \mathbf{H}_{ik}, \quad j, k = 1, \ldots, n$$

We define the small oscillations of points P_i relative to body G by the following Lagrange equations:

$$Aq^{\cdot\cdot} + Bq^{\cdot} + Cq = Q \tag{1.5}$$

where A, B, and C are constant symmetric square $n \times n$ matrices assumed to be positive definite. Matrices $A = \| a_{jk} \|$ (see (1.4), $B = \| b_{jk} \|$, and $C = \| c_{jk} \|$ define, respectively, the kinetic energy, the dissipation, and the potential elastic energy. In Eqs. (1.5) Q denotes the n-dimensional vector of generalized forces $Q_1, \ldots, Q_n$ which are due to inertia and external forces $\mathbf{F}_i$ acting on points P_i in the coordinate system $Ox_1x_2x_3$. They are obtained from the latter forces by the transformation contravariant to the substitution (1.3) (see [7]) and are of the form

$$Q_j = \sum_{i=1}^{N} \mathbf{H}_{ij} \{\mathbf{F}_i - m_i [\mathbf{R}_0^{\cdot\cdot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\boldsymbol{\rho}_i + \mathbf{r}_i)) + \boldsymbol{\omega}' \times (\boldsymbol{\rho}_i + \mathbf{r}_i) + 2\boldsymbol{\omega} \times \mathbf{r}_i']\}, \quad j = 1, \ldots, n \tag{1.6}$$

The external forces $\mathbf{F}_i$ that act on each point P_i are assumed to depend on the absolute position and the velocity of these

$$\mathbf{F}_i = \mathbf{F}_i(\mathbf{R}_i, \mathbf{R}_i^{\cdot}, t) = \mathbf{F}_i(\mathbf{R}_0 + \boldsymbol{\rho}_i + \mathbf{r}_i, \quad \mathbf{R}_0^{\cdot} + \boldsymbol{\omega} \times (\boldsymbol{\rho}_i + \mathbf{r}_i) + \mathbf{r}_i', t), \quad i = 1, \ldots, n \tag{1.7}$$

in the derivation of which formulas (1.1) and (1.2) and the equalities $\rho_i' = 0$, $i = 1, \ldots, n$ were used.

Moments about pole O of the complete system S moving in relation to the coordinate system $OX_1X_2X_3$ are defined by the equation

$$\mathbf{K}^{\cdot} = \mathbf{M} + \mathbf{M}_1 \tag{1.8}$$

where $\mathbf{K}$ is the moment of momentum of system S relative to pole O in its motion relative to that coordinate system, and $\mathbf{M}$ and $\mathbf{M}_1$ are the principal moments of all external and inertia forces, respectively, acting on system S. By definition these quantities are

$$\mathbf{K} = \sum_\alpha m_\alpha (\boldsymbol{\rho}_\alpha + \mathbf{r}_\alpha) \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_\alpha + \mathbf{r}_\alpha^{\cdot}) \tag{1.9}$$

$$\mathbf{M} = \sum_\alpha (\boldsymbol{\rho}_\alpha + \mathbf{r}_\alpha) \times \mathbf{F}_\alpha, \quad \mathbf{M}_1 = -\sum_\alpha m_\alpha (\boldsymbol{\rho}_\alpha + \mathbf{r}_\alpha) \times \mathbf{R}_0^{\cdot\cdot}$$

Formulas (1.1) and (1.2) were used here, and summation was taken over all points $\mathbf{R}_\alpha$ of system S. For points of the solid body G it is necessary to set $\mathbf{r}_\alpha \equiv 0$ in (1.9).

For the subsequent analysis it is convenient to introduce the subsidiary system S^* consisting of the solid body G and points P_i rigidly fixed in their equilibrium positions O_i. Mass of the solid body S^* is

$$m^* = m + \sum_{i=1}^{N} m_i \tag{1.10}$$

and for it all $\mathbf{r}_i = 0$. We denote the center of mass of body S^* by C and its inertia tensor relative to point O by $\mathbf{J}$. Tensor $\mathbf{J}$ is obviously constant in the coordinate system $Ox_1x_2x_3$. The quantities (1.9) can be represented in the form

$$\mathbf{K} = \mathbf{J} \cdot \boldsymbol{\omega} + \sum_{i=1}^{N} m_i [\mathbf{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i + \mathbf{r}_i^{\cdot}) + \boldsymbol{\rho}_i \times \mathbf{r}_i^{\cdot}] \tag{1.11}$$

$$\mathbf{M} = \mathbf{M}^* + \sum_{i=1}^{N} [\mathbf{r}_i \times \mathbf{F}_i + \boldsymbol{\rho}_i \times (\mathbf{F}_i - \mathbf{F}_i^*)]$$

$$\mathbf{M}_1 = \mathbf{M}_1^* - \sum_{i=1}^{N} m_i \mathbf{r}_i \times \mathbf{R}_0^{\cdot\cdot}$$

Here and subsequently the asterisk denotes quantities that relate to the solid body S^*, i.e. calculated with $\mathbf{r}_i \equiv 0,\ i = 1, \ldots, N$.

For the moments $\mathbf{M}^*$ and $\mathbf{M}_1^*$ we have

$$\mathbf{M}^* (\mathbf{R}_0, \mathbf{R}_0^{\cdot}, \boldsymbol{\omega}, \sigma, t) = \sum_\alpha \boldsymbol{\rho}_\alpha \times \mathbf{F}_\alpha^* \tag{1.12}$$

$$\mathbf{M}_1^* = -\sum_\alpha m_\alpha \boldsymbol{\rho}_\alpha \times \mathbf{R}_0^{\cdot\cdot}$$

Moment $\mathbf{M}^*$ may depend on variables that define motion of the body S^*, i.e. on $\mathbf{R}_0, \mathbf{R}_0^{\cdot}, \boldsymbol{\omega}, t$, and on the vector parameter σ which defines the orientation of system $Ox_1x_2x_3$ relative to system $OX_1X_2X_3$. Euler's angles or the directional cosines of system $Ox_1x_2x_3$ relative to system $OX_1X_2X_3$ may be taken as components of vector σ. Vector σ satisfies the conventional equations (Euler's kinematic equations) for a solid body, which we write in the simplified form

$$\sigma^{\cdot} = f(\sigma, \boldsymbol{\omega}) \tag{1.13}$$

Let the motion of point O be specified, i.e. let $\mathbf{R}_0(t)$ be a known function of time. This is, for instance, so when body G has a fixed point. The motion of system S is then completely determined by Eqs. (1.5), (1.8), and (1.13) and formulas (1.1) - (1.4), (1.6), (1.7), (1.11), and (1.12). In that case the moment

$\mathbf{M}_1^*$ of inertia forces in (1.11) depends only on the orientation of σ and on time t as defined by $\mathbf{R}_0^{\cdot\cdot}(t)$.

If, however, the motion of point O is not specified, these equations must be supplemented by the equation of momentum of system S. In that case it is convenient to take the center of mass C of body S^* as the pole O. Then taking into account the second of equalities (1.12), we obtain

$$\sum_\alpha m_\alpha \rho_\alpha \equiv 0, \quad \mathbf{M}_1^* = 0 \tag{1.14}$$

The equation of momentum change then assumes the form

$$m^*\mathbf{R}_0^{\cdot\cdot} = \mathbf{F}^* + \sum_{i=1}^{N} (\mathbf{F}_i - \mathbf{F}_i^* - m_i \mathbf{r}_i^{\cdot\cdot}) \tag{1.15}$$

where m^* is defined by formula (1.10) and $\mathbf{F}^* = \mathbf{F}^*(\mathbf{R}_0, \mathbf{R}_0^{\cdot}, \sigma, \omega, t)$ is the principal vector of all external forces acting on the solid body S^*, i.e. when $\mathbf{r}_i \equiv 0$, $i = 1, \ldots, N$. The derived equations of motion will be analyzed and simplified below.

2. We introduce in the analysis three characteristic time scales: the characteristic period T_1 of free elastic oscillations of points P_i relative to body G, the characteristic time T_2 of damping of such oscillation, and the characteristic time T_3 of motion of system S as a whole. For instance, we can assume that $T_3 \sim \omega^{-1}$. The above time scales are assumed to satisfy the inequalities

$$T_1 \ll T_2 \ll T_3 \tag{2.1}$$

If conditions (2.1) are satisfied, time T_2 of free elastic oscillation damping is considerably shorter than time T_3 of rotation of the body about its center of mass. Hence in analyzing the evolution of the system motion during time intervals of order T_3' and longer it is possible to neglect the free oscillations and consider only forced motions of points P_i induced by external and inertia forces. To satisfy conditions (2.1) we set in Eqs. (1.6)

$$C = \varepsilon^{-2}C^\circ, \quad B = \delta\varepsilon^{-1}B^\circ \tag{2.2}$$

where C° and B° are matrices with bounded elements, and ε and δ are small dimensionless parameters that satisfy conditions

$$0 < \varepsilon \ll \delta \ll 1 \tag{2.3}$$

In the limit case of $\varepsilon \to 0$ which corresponds to infinitely great rigidity of elastic links equalities (1.5) and (2.2) imply that $q \equiv 0$, and from formulas (1.3) we then obtain $\mathbf{r}_i \equiv 0$ for $i = 1, \ldots, N$. Thus, when $\varepsilon = 0$, Eqs. (1.8), (1.11), and (1.15) with allowance for (1.2) yield

$$\mathbf{J}\cdot\omega' + \omega \times (\mathbf{J}\cdot\omega) = \mathbf{M}^* + \mathbf{M}_1^*, \quad m^*\mathbf{R}_0^{\cdot\cdot} = \mathbf{F}^* \tag{2.4}$$

which are the equations of motion of the solid body S^* into which system S is transformed when $\varepsilon \to 0$.

For small positive ε and δ Eq. (1.5) with conditions (2.2) assumes the form

$$\varepsilon^2 A q^{\cdot\cdot} + \delta\varepsilon B^\circ q^{\cdot} + C^\circ q = \varepsilon^2 Q \qquad (2.5)$$

An approximate solution of Eq. (2.5) with small parameters (2.3) at derivatives can be obtained by asymptotic methods (see, e.g., [8]). It consists of a regular part and of a solution of the type of boundary layer that is rapidly damped as the time from the initial instant inscreases.

Let us begin by considering the free elastic oscillations defined by the homogeneous system (2.5) for $Q = 0$. The related characteristic equation is of the form

$$\det\,(\varepsilon^2\lambda^2 A + \delta\varepsilon\lambda\, B^\circ + C^\circ) = 0 \qquad (2.6)$$

We pass from generalized coordinates q to conventional ones in which the two positive definite matrices A and C° simultaneously reduce to a diagonal form [7]. Such transformation reduces matrix A to unit matrix I, matrix C° to the diagonal matrix C_1° with positive diagonal elements, and matrix B° to some positive definite matrix B_1°. The characteristic equation (2.6) then becomes

$$\det\,(\Lambda^2 I + \delta\Lambda B_1^\circ + C_1^\circ) = 0, \quad \Lambda = \varepsilon\lambda \qquad (2.7)$$

Roots of Eq. (2.7) are determined by expansions in the small parameter δ of the form

$$\Lambda_j = \pm\, i\,(C_1^\circ)_{jj} - {}^1/_2\delta\,(B_1^\circ)_{jj} + O\,(\delta^2), \quad j = 1, \ldots, n \qquad (2.8)$$

where subscripts jj denote the diagonal elements of matrices; these elements are positive. Reverting to variable λ in (2.7) from (2.8) we obtain

$$\lambda_j = \pm\, i\Omega_j - {}^1/_2\delta\varepsilon^{-1}(B_1^\circ)_{jj} + O\,(\delta\varepsilon^{-1}) \qquad (2.9)$$

$$\Omega_j = \varepsilon^{-1}(C_1^\circ)_{jj}, \quad j = 1, \ldots, n$$

Quantities Ω_j are the natural oscillation frequencies of the conservative system to which system (1.5) reduces when $B = 0$. Equalities (2.9) yield the estimates

$$T_1 = O\,(\varepsilon^{-1}), \;\; T_2 = O\,(\varepsilon^{-1}\delta), \quad T_3 = O\,(1) \qquad (2.10)$$

the last of which follows from the independence of T_3 from parameters ε and δ. It follows from formulas (2.10) and (2.3) that inequalities (2.1) are satisfied. The free oscillations that correspond to eigenvalues of (2.9) represent the rapidly damped part of solution of the type of boundary layer. Some time after the initial instant, i.e. for times of order of T_3 and greater, such oscillations can be disregarded.

That part of solution of system (2.5) that is regular with respect to ε and δ is obtained in the form of expansions in powers of parameters ε^2 and $\varepsilon\delta$. Allowing for inequalities (2.3) we obtain

$$q = \varepsilon^2 q^{(0)} + \varepsilon^3\delta q^{(1)} + O\,(\varepsilon^4) \qquad (2.11)$$

Substituting expansions (2.11) into Eq. (2.5) and equating coefficients at powers of parameters ε and δ, we obtain

$$q^{(0)} = (C^0)^{-1}Q^*, \quad q^{(1)} = -(C^0)^{-1}B^0 q^{(0)\cdot} \qquad (2.12)$$

where Q^* is the vector of generalized forces in which it is necessary to set $q \equiv 0$. Consequently, forces Q_j^* relate to the solid body S^* and are determined by formulas (1.6) and (1.7) for $r_i \equiv 0,\ i = 1, \ldots, N$. We have

$$Q_j^* = \sum_{i=1}^{N} \mathbf{H}_{ij} \{\mathbf{F}_i^* - m_i [\mathbf{R}_0^{\cdot\cdot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) + \boldsymbol{\omega}' \times \boldsymbol{\rho}_i]\} \tag{2.13}$$

$$\mathbf{F}_i^* = \mathbf{F}_i (\mathbf{R}_0 + \boldsymbol{\rho}_i, \mathbf{R}_0^{\cdot} + \boldsymbol{\omega} \times \boldsymbol{\rho}_i, t); \quad i = 1, \ldots, N; \quad j = 1, \ldots, n$$

Using the notation in (2.2) we write the solution of (2.11), (2.12) as

$$q = C^{-1} [Q^* - BC^{-1} (Q^*)^{\cdot}] + O(\varepsilon^4) \tag{2.14}$$

This solution defines the small forced motions of points P_i relative to body G.

3. To simplify the equations of motion of system S on assumptions stated above it is necessary to substitute solution (2.14) into formulas (1.8), (1.11), and (1.15). Note that according to (2.2) vector q in (2.14) is of order ε^2. The order of vectors $\mathbf{r}_i$ defined by formula (1.3) is the same. Taking this into consideration we reduce Eqs. (1.8), (1.11), and (1.15) to the form

$$\mathbf{J} \cdot \boldsymbol{\omega}' + \boldsymbol{\omega} \times (\mathbf{J} \cdot \boldsymbol{\omega}) = \mathbf{M}^* + \mathbf{M}_1^* + \boldsymbol{\mu}, \quad m^* \mathbf{R}_0^{\cdot\cdot} = \mathbf{F}^* + \boldsymbol{\varphi} \tag{3.1}$$

$$\boldsymbol{\mu} = \sum_{i=1}^{N} \Big\{ \mathbf{r}_i \times \mathbf{F}_i^* + \boldsymbol{\rho}_i \times \Big(\frac{\partial \mathbf{F}_i^*}{\partial \mathbf{R}_i} \cdot \mathbf{r}_i + \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{R}_i^{\cdot}} \cdot \mathbf{r}_i^{\cdot} \Big) - \tag{3.2}$$

$$m_i (\mathbf{r}_i \times \mathbf{R}_0^{\cdot\cdot}) - m_i [\mathbf{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) + \boldsymbol{\rho}_i \times \mathbf{r}_i^{\cdot}]^{\cdot} \Big\} + O(\varepsilon^4)$$

$$\boldsymbol{\varphi} = \sum_{i=1}^{N} \Big(\frac{\partial \mathbf{F}_i^*}{\partial \mathbf{R}_i} \cdot \mathbf{r}_i + \frac{\partial \mathbf{F}_i^*}{\partial \mathbf{R}_i^{\cdot}} \cdot \mathbf{r}_i^{\cdot} - m_i \mathbf{r}_i^{\cdot\cdot} \Big) + O(\varepsilon^4)$$

where $\partial \mathbf{F}_i^* / \partial \mathbf{R}_i$ and $\partial \mathbf{F}_i^* / \partial \mathbf{R}_i^{\cdot}$ are matrices of partial derivatives of funtions $\mathbf{F}_i^*$ (2.13) with respect to components of their vector arguments.

Equations (3.1) are similar to the equations of motion (2.4) for the solid body S^*, except for the terms $\boldsymbol{\mu}$ and $\boldsymbol{\varphi}$. These terms may be considered to be the principal moment about point O and the principal vector, respectively, of forces acting on the solid S^* and produced by its elastic and dissipative elements. Vectors $\boldsymbol{\mu}$ and $\boldsymbol{\varphi}$ in (3.2) are linearly dependent of vectors $\mathbf{r}_i$ and their derivatives, and contain terms of order ε^2 and $\varepsilon^3\delta$. Terms $O(\varepsilon^2)$ and $O(\varepsilon^3\delta)$ correspond, respectively, to internal elastic and dissipation forces.

We shall show that vectors $\boldsymbol{\mu}$ and $\boldsymbol{\varphi}$ can be defined with an accuracy within quantities of order ε^4 in terms of variables $\mathbf{R}_0$, $\mathbf{R}_0^{\cdot}$, σ, $\boldsymbol{\omega}$, and t only that define the motion of the solid S^*. For this we first substitute in (3.2) for the derivatives $\mathbf{r}_i^{\cdot}$ and $\mathbf{r}_i^{\cdot\cdot}$ their expressions in formula (1.2) and, then, use formulas (1.3) for representing vectors $\mathbf{r}_i$, $\mathbf{r}_i'$ and $\mathbf{r}_i''$ in terms of q, $q^{\cdot}$, and $q^{\cdot\cdot}$. (When differentiating (1.3) it is necessary to take into account that in the coordinate system $Ox_1x_2x_3$ vectors $\mathbf{H}_{ij}$ are constant). After this vectors $\boldsymbol{\mu}$ and $\boldsymbol{\varphi}$ depend only on vector q and its derivatives, which we eliminate using formulas (2.14) and (2.13). As the result, vectors $\boldsymbol{\mu}$ and $\boldsymbol{\varphi}$ represented by functions of vectors $\mathbf{R}_0$ and $\boldsymbol{\omega}$ and of their derivatives, and also on vectors $\mathbf{H}_{ij}$ and $\boldsymbol{\rho}_i$ which in the coordinate system $Ox_1x_2x_3$ are known constants. The expressions for $\boldsymbol{\mu}$ and $\boldsymbol{\varphi}$ contain higher derivatives of vectors $\mathbf{R}_0$ and $\boldsymbol{\omega}$, hence Eqs. (3.1) are of an order that is formally higher than that of conventional equations of solid body dynamics. However these higher derivatives may be excluded without loss of accuracy.

Note that the equations of motion (2.4) of the solid S^* can always be solved for higher derivatives, i.e. for ω' and $\mathbf{R}_0^{\cdot\cdot}$, which yields formulas

$$\omega' = f_1(\mathbf{R}_0, \mathbf{R}_0^{\cdot}, \sigma, \omega, t), \quad \mathbf{R}_0^{\cdot\cdot} = f_2(\mathbf{R}_0, \mathbf{R}_0^{\cdot}, \sigma, \omega, t) \tag{3.3}$$

Differentiating equalities (3.3) with allowance for (1.13), we can obtain formulas for higher derivatives ω'', $\mathbf{R}_0^{\cdot\cdot\cdot}$, etc. in terms of the same variables $\mathbf{R}_0$, $\mathbf{R}_0^{\cdot}$, σ, ω, and t. Since the derived equations of motion (3.1) of a deformable body differ from the equations of motion (2.4) of a solid by the terms μ and φ of order ε^2, expressions (3.3) and their derivatives are also valid for Eqs. (3.1) with an error of $O(\varepsilon^2)$. Hence the substitution of expressions (3.3) and their derivatives into formulas (3.2) for the quantities μ and φ, which are themselves of order ε^2, results in an error $O(\varepsilon^4)$ which is within the accuracy of equalities (3.2).

Eliminating derivatives ω', ω'', $\mathbf{R}_0^{\cdot\cdot}$, $\mathbf{R}_0^{\cdot\cdot\cdot}$, etc., using equalities (3.3) and their derivatives, we thus obtain the sought functions of the form

$$\mu = \mu(\mathbf{R}_0, \mathbf{R}_0^{\cdot}, \sigma, \omega, t), \quad \varphi = \varphi(\mathbf{R}_0, \mathbf{R}_0^{\cdot}, \sigma, \omega, t) \tag{3.4}$$

accurate within $O(\varepsilon^4)$. Functions (3.4), as moment $\mathbf{M}^*$ in (1.12) and force $\mathbf{F}^*$ in (1.15), depend only on the parameters of motion of the body S^*. Hence Eqs. (3.1) together with (3.4) and (1.13) constitute a closed system similar to that of equations of motion of a solid body. These equations define the evolution of motions of the deformable system S through time intervals that are considerably longer than the time of natural elastic oscillation damping. The quantities μ and φ represent here small perturbations, hence it is possible to apply to the derived system various methods of the small parameter, in particular the method of averaging.

Functions (3.4) are not presented here in their explicit form owing to their unwieldiness, however, the transformation procedure described above, which uses formulas (3.2), (1.2), (1.3), (2.13), (2.14), (2.4) and (3.3) makes it possible to uniquely devise functions (3.4).

4. Two examples of determination of the structure of function (3.4) are presented below. In both of these it is assumed that moment $\mathbf{M}^*$ and all external forces $\mathbf{F}_i$ acting on points P_i are zero. In conformity with (2.13) and (3.2) we then have

$$Q_j^* = -\sum_{i=1}^{N} \mathbf{H}_{ij} m_i [\mathbf{R}_0^{\cdot\cdot} + \omega \times (\omega \times \rho_i) + \omega' \times \rho_i], \quad j = 1, \ldots, n \tag{4.1}$$

$$\mu = -\sum_{i=1}^{N} m_i \{\mathbf{r}_i \times \mathbf{R}_0^{\cdot\cdot} + [\mathbf{r}_i \times (\omega \times \rho_i) + \rho_i \times \mathbf{r}_i^{\cdot}]^{\cdot}\}, \quad \varphi = -\sum_{i=1}^{N} m_i \mathbf{r}_i^{\cdot\cdot}$$

In the first example we furthermore assume that point O is stationary, $\mathbf{R}_0 \equiv 0$, which in accordance with (1.12) implies that $\mathbf{M}_1^* = 0$.

In the second example the complete system is assumed free of external forces, i.e. $\mathbf{F}^* = 0$. Taking the center of mass C of system S^* as the pole O, we have in accordance with (1.14) $\mathbf{M}_1^* = 0$.

Thus in both examples $\mathbf{M}^* = \mathbf{M}_1^* = 0$, and the first of Eqs. (3.1) yields

$$\mathbf{J}\cdot\boldsymbol{\omega}' + \boldsymbol{\omega}\times(\mathbf{J}\cdot\boldsymbol{\omega}) = \boldsymbol{\mu} = O(\varepsilon^2) \tag{4.2}$$

and the first of equalities (3.3) assumes the form

$$\boldsymbol{\omega}' = -\mathbf{J}^{-1}\cdot(\boldsymbol{\omega}\times\mathbf{J}\cdot\boldsymbol{\omega}) + O(\varepsilon^2) \tag{4.3}$$

Differentiation of equality (4.3) shows that the k-th derivative $\boldsymbol{\omega}^{(k)}$ is a homogeneous polynomial of $k+1$ power of components of vector $\boldsymbol{\omega}$, with $k = 0, 1, \ldots$. accurate within terms of order ε^2 .

In the first example $(\mathbf{R}_0 \equiv 0)$ the first of equalities (4.1) shows that Q_j^* are homogeneous polynomials in second powers of $\boldsymbol{\omega}$ of order $m_0 l\omega^2$, where m_0 is the characteristic mass of points P_i and l is a characteristic linear dimension of order ρ_i. Then from equality (2.14) follows that q is the sum of homogeneous polynomials in second and third powers of $\boldsymbol{\omega}$ of orders $m_0 lc^{-1}\omega^2$ and $m_0 lc^{-2}b\omega^3$, respectively. Here c is the characteristic stiffness of elastic links (a quantity of the order of elements of matrix C) and b is the characteristic dissipation coefficient (a quantity of the order of elements of matrix B). The structure of vectors $\mathbf{r}_i$, $i = 1, \ldots, N$ which are determined by formula (1.3) is the same. Note that (4.3) implies that each differentiation raises the power of polynomials in $\boldsymbol{\omega}$ by one. Hence, for $R_0 \equiv 0$ vector $\boldsymbol{\mu}$ in (4.1) is the sum of homogeneous polynomials in the fourth and fifth powers of components ω_j of vector $\boldsymbol{\omega}$, namely,

$$\boldsymbol{\mu} = \boldsymbol{\mu}_4(\boldsymbol{\omega}) + \boldsymbol{\mu}_5(\boldsymbol{\omega}) + O(\varepsilon^4) \tag{4.4}$$

$$\boldsymbol{\mu}_4(\boldsymbol{\omega}) = \sum_{j,k,l,m=1}^{3} \mathbf{D}_{jklm}\omega_j\omega_k\omega_l\omega_m = O\left(\frac{m_0 l^2\omega^4}{c}\right) = O(\varepsilon^2)$$

$$\boldsymbol{\mu}_5(\boldsymbol{\omega}) = \sum_{j,k,l,m,n=1}^{3} \mathbf{E}_{jklmn}\omega_j\omega_k\omega_l\omega_m\omega_n = O\left(\frac{m_0 lb\omega^3}{c^2}\right) = O(\varepsilon^3\delta)$$

where the orders of quantities correspond to those in (2.2). In the coordinate system attached to the solid body the coefficients $\mathbf{D}_{jklm}$ and $\mathbf{E}_{jklmn}$ are constants expressed in terms of constants m_j, $\mathbf{J}$, C°, B°, $\mathbf{H}_{ij}$, and ρ_i. Polynomials $\boldsymbol{\mu}_4$ and $\boldsymbol{\mu}_5$ represent the moments of elastic and dissipation forces, respectively.

Let us pass to the second example $(\mathbf{F}^* = 0)$. The second of Eqs. (3.1) implies that in this case $\mathbf{R}_0^{\cdot\cdot} = O(\varepsilon^4)$. Hence the quantities Q_j^* and $\boldsymbol{\mu}$ are in (4.1) of the same form(with the accepted accuracy $O(\varepsilon^4)$) as in the first example. The perturbing moment $\boldsymbol{\mu}$ is again defined by formulas (4.4), and the perturbing force $\boldsymbol{\varphi}$ in (4.1) by similar formulas in the form of the sum of homogeneous polynomials in the fourth and fifth powers of $\boldsymbol{\omega}$.

Equations (4.2) and (4.1) were originally derived in [5] for the case of a single point P_i on an elastic link $(N = 1)$.

Formulas (4.4) were obtained in explicit form for some particular cases (of symmetrical solid body S^*), and Eq. (4.2) with moment defined by (4.4) was integrated in [5].

5. Similar analysis can be applied to a solid body to which instead of discrete points are attached solid elastic bodies such as rods, plates, or shells, with linear dissipation. However in that case it is necessary to use instead of solution (2.14) the related quasi-static solution of equations of elastic equilibrium under the action of external and inertia forces. It can be shown that equations of the form (3.1) and (3.4), as well as Eqs. (4.2) and (4.4) which are particular cases of these, remain valid. Since in the case of a free system inertia forces (4.1) are quadratic forms of ω, elastic translations are also proportional to the square of components of vector ω. Repeating the reasoning of Sect. 4 we obtain the formulas of the form (4.4). As previously, the condition of validity of Eqs. (3.1) and (3.4), or (4.2) and (4.4), is of the form (2.1) in which T_1 is the greatest period of natural elastic oscillations, T_2 is the characteristic time of their damping, and $T_3 \sim \omega^{-1}$ is the characteristic time of motion of the system as a whole. Taking $\Omega \sim T_1^{-1}$ as the minimum natural elastic oscillation frequency of the elastic solid body, we can represent inequalities (2.1) in one of the following forms:

$$\Omega \gg T_2^{-1} \gg \omega, \quad V \gg lT_2^{-1} \gg v$$

where l is a characteristic linear dimension of the system, $V \sim \Omega l$ is the characteristic velocity of elastic waves, and $v \sim \omega l$ is the characteristic linear velocity of the system rotation.

Note that the theory presented here does not cover the phenomenon of resonance between the rotation of the body itself and one of the natural elastic oscillations.

6. Let us compare Eqs. (4.2) and (4.4) with the results obtained earlier [9] for the motion of a solid body with a cavity containing a viscous fluid at various Reynolds numbers. Let, for example, that number be small, $\mathrm{Re} = \omega l^2 \nu^{-1} \ll 1$, where ω is the angular velocity of the body, l is a characteristic linear dimension of the cavity, and ν is the fluid kinematic viscosity. Equations of motion of the body relative to its center of mass can then be reduced to the form (4.2) [9], with vector $\mu(\omega)$ in the form of a homogeneous third power polynomial in ω whose coefficients depend on the form of the cavity. A similar result was obtained in [9] for a free gyrostat with flywheels or balls which could rotate inside it under the action of viscous friction forces between the body and the rotating masses. Thus the energy dissipation by the process of internal elastic oscillations presents a similar, but more complex, pattern of the evolution of body motion relative to its center of mass than that of energy dissipation by viscous fluid in the cavity of a solid body at high Reylolds numbers.

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